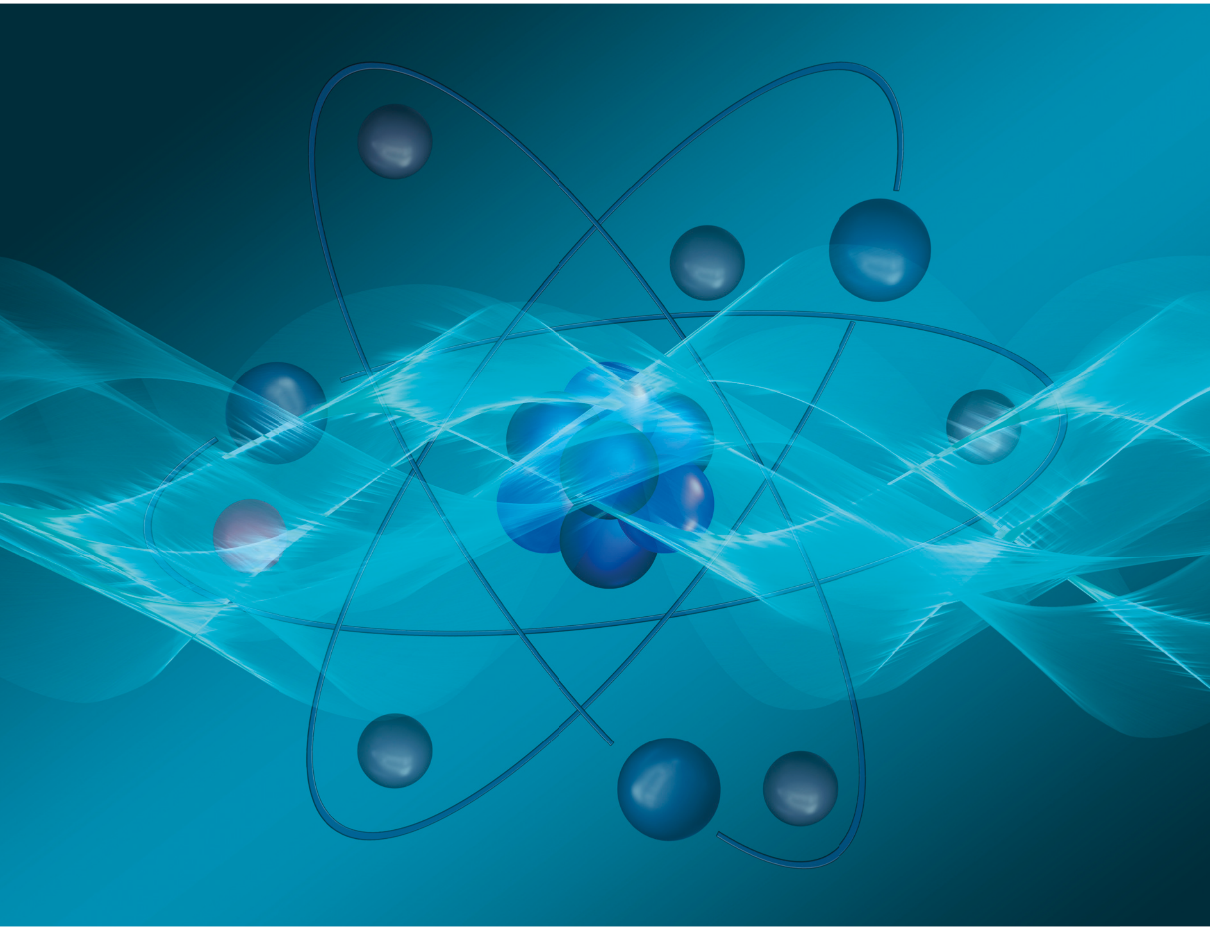
JIŘÍ HOŘEJŠÍ

Lectures on
QUANTUM FIELD THEORY



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Jiří Hořejší

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Charles University
Karolinum Press

www.karolinum.cz
ebooks@karolinum.cz

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info@matfyzpress.cz

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Preface

This work covers the material of a two-semester course of quantum field theory (QFT) that I taught for more than 20 years at the Charles University and Czech Technical University in Prague. For years, I was reluctant to write up such a set of lecture notes, since the current literature in this area is quite rich and there are dozens of books on the subject. However, eventually I was forced to do it, because of the pandemy of the infamous coronavirus that has broken out in spring 2020. I comment on this in more detail below. Conceptually, my approach is traditional, starting with several introductory chapters on the relativistic quantum mechanics. Then, after a brief interlude on the classical field theory, one proceeds to the quantization of free fields and to some elementary examples of field interactions, the basic tool being the Dyson perturbation expansion of the S-matrix in the interaction representation. The pragmatic aim of the first half of the text (chapters 1–25) is to arrive at the basic techniques for calculations of Feynman diagrams in the lowest perturbative order, as well as for the computation of the particle decay rates and scattering cross sections. This is just the matter that should be ideally explained during the first (winter) semester, since a part of the curriculum in the second (summer) semester, at least for some students, is a course on the standard model of particle physics, where a Feynman diagram calculation is an everyday occurrence. The second half (chapters 26–50) represents topics to be explained during the second semester and the main theme here is quantum electrodynamics at the level of one-loop diagrams, including techniques of regularization of ultraviolet divergences and renormalization. In this way, the whole material of the present lecture notes is divided into 50 chapters and each of them corresponds, roughly, to a 90 min. lecture (the total number of QFT lectures in a given academic year is about fifty). I would like to stress that the text is really intended to have the character of lecture notes, which means that, among other things, some explicit calculations are shown here in greater detail than in most of the representative monographs and textbooks, so as to make the life of a QFT beginner easier. Throughout the text one also encounters numerous hints to possible independent calculations, addressed to interested diligent readers; some of the problems in question may also serve as appropriate topics for tutorials. Admittedly, readers that are not quite fond of performing independent calculations may find the repeated offers of problems left to them as "instructive exercises" somewhat disturbing (or even annoying); anyway, there are just about three dozen of such hints in the whole text, i.e. less than one per chapter on average.

As I have indicated above, these lecture notes have been written under rather special circumstances, during the protracted coronavirus (COVID-19) crisis in 2020 and 2021. It was a situation that people of my generation have experienced never before, so let me add some personal recollection (which is, admittedly, somewhat emotional). The outbreak of the pandemy was officially announced in March 2020. Thus, on Wednesday, March 11, the personal attendance of students in the lecture rooms was banned "until further notice" and I decided to write immediately the text of a lecture scheduled for Thursday, to be able to send it to students via e-mail. Such a procedure seemed to me more efficient than a system of videoconferences or so, and I hoped also that the students' opinion would coincide with that of the aspiring student in Goethe's Faust, expressed in a dialogue with Mephistopheles, namely, "You won't need to tell me twice! I think, myself, it's very helpful, too, that one can take back home, and use, what someone's penned in black and white".[1] In any case, it is obvious that a carefully written text is more durable than lectures presented on a blackboard and erased immediately after the classes. Thus I went on in this manner, sticking to the maxim "nulla dies sine linea", till the end of May when the semester terminates. When the summer semester and the students' exams were over, I returned to the material of the envisaged next winter semester and continued writing down the relevant lectures so as to have a complete set (in musical terms, "da capo al fine"). In the meantime, I had to put together a collection of lectures for another course, aimed at a more advanced audience (25 chapters as well). In this way, the whole work has been basically completed in May 2021, with the nasty virus still around. Then there followed a period of transforming the manuscript full of handwritten formulae into a user-friendly electronic file, as well as gradual detailed proofreading of the text, mostly during the academic year 2021/2022. This was largely finished in autumn 2022, when the pandemy was fading away, but was overshadowed by even more tragic events — of course, I have in mind the absurd criminal war that Russia started against Ukraine.

When I started writing the lecture notes, in the gloomy atmosphere of the covid calamity on the rise, it came to my mind that there is a famous work of the world literature that was created under similar circumstances and survived over centuries. Yes, you guessed right; it is the Decameron by Giovanni Boccaccio. Its origin is widely known. It represents a collection of one hundred tales told by a group of ten young people who escaped from Florence, where the epidemic of plague broke out in 1348, and stayed in a hideout in the countryside to avoid the dangerous infection. Concerning my text, I have also written the lecture notes partly in a hideout (the "home office"). These consist of only fifty tales told by myself (not young anymore), concerning topics not so easily accessible to a general public and I certainly do not expect that my opus will become so famous as the Boccaccio's Decameron, or that it could survive through centuries. Nevertheless, I believe that it may have an appropriate (though inevitably limited) lifetime and may be useful for at least some students and other potentially interested scientifically minded readers. My primary aim has been to make it a comprehensible and digestible introduction to the rather difficult subject of quantum field theory, which, among others, forms a basis of the contemporary particle physics.

One last remark is perhaps in order here. In view of the above-mentioned origin of these lecture notes, it is to be expected that most of the potential readers will be university students fluent in Czech. Thus, I could not resist the temptation to include, occasionally, some notes concerning the Czech equivalents of the international English terminology, or even some elements of a common literary folklore. Hopefully, this might add some cheering moments to the serious scholarly style of the whole opus.

Acknowledgements

From what I have written above it might seem that I should thank the malicious coronavirus in the first place, for stimulating me to write up these lecture notes. But I will not, taking into account that, apart from the positive impact mentioned above, this dangerous invisible bug did also so much harm to so many people all over the world. Needless to say, my acknowledgements are aimed in a completely different, genuinely positive, direction. In particular, I recognize the work of my younger colleagues who conducted and supervised, during the previous years, the

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[1] A translation into English by A. S. Kline, 2003. In Czech (in the classic translation by O. Fischer) it reads: "Tot' praktické, i heleď'me se! To tělem duší při tom jsem. Neb co je černé na bílém, to vesele se domů nese."

tutorials related to my lectures. They are, in alphabetical order: Karol Kampf, Karel Kolář, Jiří Novotný and Martin Zdráhal. Further, I appreciate questions and comments that the students made throughout the years; this certainly led to many improvements of the style and contents of the lectures. Actually, I have also received some useful remarks from other colleagues; for instance, Walter Grimus from Vienna University has drawn my attention to the fact that the frequently cited "Lorentz condition" in electromagnetism is in fact "Lorenz condition". Finally, my great thanks are due to Tomáš Husek and Tomáš Kadavý, who recast my manuscript in LaTeX and thus made it ready for publication; the whole work matured to its present form in spring 2023.

Prague, May 2023 J. Hořejší

Conventions, notations and units

Unless stated otherwise, we use the natural system of units, in which $\hbar = c = 1$ (note that Peskin and Schroeder call it "God-given" units in their book [14]). Obviously, within such a system, the time and length have the same dimension, the energy, momentum and mass have the same dimension, inverse length has the dimension of a mass, etc. The passage from the economical natural system to ordinary units is quite straightforward. To this end, one may use the commonly known approximate values of the Planck constant $\hbar$ and the "conversion constant" $\hbar c$, namely

$$\hbar = 6.58 \times 10^{-22}\,\mathrm{MeV\,s}\,, \qquad \hbar c = 197\,\mathrm{MeV\,fm}\,,$$

where $1\,\mathrm{fm} = 10^{-13}\,\mathrm{cm}$ (fm stands for "fermi" or "femtometer"). Numerical values of observable quantities (such as decay rates or scattering cross sections) are then converted into ordinary units by setting

$$1\,\mathrm{MeV}^{-1} = 6.58 \times 10^{-22}\,\mathrm{s}\,,$$

or

$$1\,\mathrm{MeV}^{-1} = 197\,\mathrm{fm}\,.$$

While the natural system of units is universally accepted in the literature concerning quantum field theory and particle physics, there are three other conventions that may differ in various books, so one must emphasize what is our particular choice (to avoid any misunderstanding when comparing our results with other books or papers). First, the metric of the flat spacetime used throughout the present text is defined by

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,.$$

In other words, the metric we are employing here has the signature $(+\,-\,-\,-)$. Let us remark that such a choice seems to be prevalent in current literature; for instance, among the books that we cite in the list of relevant literature, only [13] and [18] use the metric with the inverse signature $(-\,+\,+\,+)$. Anyway, one should keep in mind that there is no question of which metric is "right" or "wrong"; its choice is just a matter of convention. Note also that readers specialized mostly in relativity and gravitation should not worry about our notation $g_{\mu\nu}$ for the metric tensor, which they got used to employ for the general case of curved Riemann space (and distinguish the case of the flat spacetime by using the symbol $\eta_{\mu\nu}$ or so). The notation used here is a common practice in the literature concerning relativistic quantum theory and particle physics, since in this area one is dealing just with flat spacetime (nevertheless, $\eta_{\mu\nu}$ is employed conventionally e.g. in the books [11, 13] or [29]).

Second, another important convention is that for the fifth Dirac gamma matrix γ_5. Here we use the definition

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,.$$

Again, this choice seems to be prevalent in the literature (note that within our list, the books [7] and [13] define γ_5 with opposite sign).

Finally, our convention for the fully antisymmetric Levi-Civita tensor is such that

$$\varepsilon_{0123} = +1 \, .$$

In this case, one must admit that this is a minority choice, since the option prevalent in current literature is $\varepsilon^{0123} = +1$ (which corresponds to the sign change in contrast to our convention). So, the reader must be careful when comparing our formulae in Appendix C and elsewhere (see in particular (C.11)) with those presented in other textbooks. Note that the convention employed here agrees with the classic books by Bjorken and Drell [1, 2].

Chapter 1

Klein–Gordon and Dirac equations: brief history

The best known equation of quantum mechanics is undoubtedly the Schrödinger equation, which for a particle moving in an external field reads

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{x})\right)\psi\,, \tag{1.1}$$

where Δ is the Laplace operator, $\Delta = \vec{\nabla}^2$, and $V(\vec{x})$ is the potential energy corresponding to an external force. The wave function $\psi = \psi(\vec{x}, t)$ has the familiar interpretation: $|\psi(\vec{x}, t)|^2$ represents the probability density for the particle localization at the point $\vec{x}$ and time t. Erwin Schrödinger published it in 1926 (and subsequently won the Nobel Prize in 1933). Let us consider first Eq. (1.1) for a free particle, i.e. for $V = 0$. There is a simple "correspondence principle" that may serve as a recipe for recovering the Schrödinger equation. Denoting the energy as E and momentum as $\vec{p}$, one may observe the correspondence

$$\begin{aligned} E &\longleftrightarrow i\hbar\frac{\partial}{\partial t}\,, \\ \vec{p} &\longleftrightarrow -i\hbar\vec{\nabla}\,, \end{aligned} \tag{1.2}$$

which leads from the usual non-relativistic relation between kinetic energy and momentum

$$E = \frac{\vec{p}^2}{2m} \tag{1.3}$$

to the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi\,. \tag{1.4}$$

Let us stress as emphatically as possible that the correspondence (1.2) does not represent a derivation of the Schrödinger equation. This cannot be derived, it can only be postulated; this is what the founding fathers of quantum theory did. The meaning of the correspondence (1.2) is that it guarantees recovering the right relation between the energy and momentum (1.3) when the operators in (1.2) act on an appropriate wave function ψ, in particular the plane wave

$$\psi(\vec{x}, t) \propto e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}\,. \tag{1.5}$$

In the same year when the non-relativistic equation (1.4) or (1.1) was postulated, a pertinent relativistic version was considered (preferably as a quantum mechanical equation for the electron). In that case, one has to use as a motivating hint the relation between the energy and momentum valid in special relativity, i.e.

$$E^2 = c^2\vec{p}^2 + m^2c^4 \,. \tag{1.6}$$

Then, using the correspondence (1.2), one gets immediately

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = \left(-\hbar^2c^2\vec{\nabla}^2 + m^2c^4\right)\psi \,, \tag{1.7}$$

and this can be recast in a more elegant form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\psi = 0 \,. \tag{1.8}$$

Now, the differential operator in Eq. (1.8) is the familiar d'Alembert operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \,, \tag{1.9}$$

and thus we end up with

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi(x) = 0 \,. \tag{1.10}$$

The equation (1.10) had been formulated in 1926 independently by several theorists: Erwin Schrödinger (who subsequently rejected it), Oskar Klein, Walter Gordon, and Vladimir Fock (or, better Fok: the name reads Фок in Russian). So, although it is apparently an equation with many parents, it is universally called the **Klein–Gordon equation**.

A remark is perhaps in order here. The constant appearing in Eq. (1.10) is the square of inverse of the Compton wavelength and one might wonder why it happens to be there, when (1.10) clearly has nothing to do with the famous Compton process (the photon scattering on a charged particle). The answer is guessed easily on dimensional grounds: the d'Alembert operator $\Box$ has, obviously, the dimension of inverse length squared, and any possible additive constant (with the same dimension) must be made of the fundamental constants of a relativistic quantum theory, i.e. c and $\hbar$ and, eventually, the relevant mass m. The combination $\hbar/mc$ is then the only possibility how to form a constant with the dimension of length (it is a refreshing simple exercise to show that such a combination of c, $\hbar$ and m is indeed unique).[2]

For convenience, let us now pass to the natural system of units with $\hbar = 1$, $c = 1$. Using the standard relativistic covariant notation, one then has

$$\left(\Box + m^2\right)\psi(x) = 0 \,, \tag{1.11}$$

with $\Box = \partial_\mu\partial^\mu$. The simplest solutions of Eq. (1.11) have the form of plane waves; for their description one may use two linearly independent exponentials

$$\begin{aligned} \psi_{(+)}(x) &= \text{const.}\, e^{-ip\cdot x} \,, \\ \psi_{(-)}(x) &= \text{const.}\, e^{ip\cdot x} \,, \end{aligned} \tag{1.12}$$

where $p \cdot x = p_0x_0 - \vec{p}\cdot\vec{x}$ (the logic of the chosen notation will become clear shortly). Inserting (1.12) into (1.11), one gets the condition

$$p^2 = m^2 \,, \tag{1.13}$$

---

[2] In fact, sticking to the traditional terminology, $\hbar/mc$ is the Compton wavelength divided by 2π.

i.e. $p_0^2 = \vec{p}^2 + m^2$. Without loss of generality, one may choose $p_0 > 0$,

$$p_0 = \sqrt{\vec{p}^2 + m^2}\,. \tag{1.14}$$

So, as expected, one recovers the correct relation between the energy and momentum of a particle with the mass m. Using the correspondence (1.2), one sees that the solution $\psi_{(+)}(x)$ describes a state with positive energy $E = p_0$ and momentum $\vec{p}$, while $\psi_{(-)}(x)$ carries negative energy $E = -p_0$ and momentum $-\vec{p}$. In any case, the four-component quantity p satisfying (1.13) is rightly called the four-momentum of the particle with the mass m. Thus we have encountered, for the first time, the problem of a wave function for the free particle with negative energy; we will see that this is a generic feature of the equations of relativistic quantum mechanics.

In fact, there is another difficulty inherent in the Klein–Gordon equation. If one wants to implement the probabilistic interpretation of the wave function ψ, one should derive first a pertinent continuity equation connecting the probability density (for particle localization) and the density of probability current. Let us first remind the reader how one proceeds in the case of non-relativistic Schrödinger equation (1.4) (we are going to use the natural system of units, i.e. set $\hbar = 1$). We have the equations for ψ and ψ^*,

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi\,, \tag{1.15}$$

$$-i\,\frac{\partial\psi^*}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi^*\,. \tag{1.16}$$

Multiplying Eq. (1.15) by ψ^* and (1.16) by ψ, and subtracting the two equations, one gets immediately

$$i\,\frac{\partial}{\partial t}\left(\psi\psi^*\right) = -\frac{1}{2m}\left(\psi^*\,\vec{\nabla}^2\psi - \psi\,\vec{\nabla}^2\psi^*\right) = -\frac{1}{2m}\vec{\nabla}\left(\psi^*\,\vec{\nabla}\psi - \psi\,\vec{\nabla}\psi^*\right).$$

Thus one obtains the familiar result

$$\frac{\partial}{\partial t}\rho_{\text{Sch.}} + \vec{\nabla}\cdot\vec{j}_{\text{Sch.}} = 0\,, \tag{1.17}$$

with

$$\rho_{\text{Sch.}} = \psi\psi^* = |\psi|^2\,, \qquad \vec{j}_{\text{Sch.}} = \frac{1}{2mi}\left(\psi^*\,\vec{\nabla}\psi - \psi\,\vec{\nabla}\psi^*\right). \tag{1.18}$$

For the Klein–Gordon equation (1.11) one may try to proceed in a similar manner. To begin with, (1.11) is recast as

$$\frac{\partial^2\psi}{\partial t^2} = \vec{\nabla}^2\psi - m^2\psi\,, \tag{1.19}$$

and the same equation holds for ψ^*. Next, using the multiplication and subtraction trick as above, one gets first

$$\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \vec{\nabla}\left(\psi^*\,\vec{\nabla}\psi - \psi\,\vec{\nabla}\psi^*\right). \tag{1.20}$$

In order to make the left-hand side of Eq. (1.20) real, one has to include a factor of i; for getting quantities with the same dimension as in the case of the Schrödinger equation, one may write finally

$$\frac{\partial}{\partial t}\rho_{\text{KG}} + \vec{\nabla}\cdot\vec{j}_{\text{KG}} = 0\,,$$

where

$$\begin{aligned} \rho_{\mathrm{KG}} &= \frac{i}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right), \\ \vec{j}_{\mathrm{KG}} &= \frac{1}{2mi}\left(\psi^* \vec{\nabla}\psi - \psi\, \vec{\nabla}\psi^*\right). \end{aligned} \tag{1.21}$$

Obviously, in contrast to (1.18), the would-be probability density ρ_{KG} in (1.21) is not *a priori* positive. In particular, it is easy to see that for $\psi_{(+)}$ shown in (1.12) the expression for ρ_{KG} is positive, while for $\psi_{(-)}$ one gets a negative value of ρ_{KG}. This, of course, is a serious flaw. On the top of that, it has soon become clear that the Klein–Gordon equation is not viable as an equation for the electron, because it cannot incorporate a description of intrinsic angular momentum, the spin (note that the concept of electron spin appeared on the physical stage in 1925, when it was introduced by George Uhlenbeck and Samuel Goudsmit — surprisingly, they have never received the Nobel Prize for it).

Despite the above-mentioned difficulty with the interpretation of the probability density, Klein–Gordon equation, as an equation of relativistic quantum mechanics, does have some limited applicability for the description of spinless particles (for more details, see e.g. the book [1]). However, we will exploit this equation fully later on, within the framework of field theory.

Anyway, it is clear that a topical question that certainly resonated in minds of quantum theorists in the second half of 1920s was: So, what is the right relativistic quantum equation for electron? The problem was resolved in 1928 by Paul Dirac. His solution was, at that time, quite astonishing and this historical breakthrough is thus worth recapitulating here (for the original paper, see [32]).

As we have already stressed, a major flaw of the Klein–Gordon equation is the non-positivity of the would-be probability density in (1.21). It is clear what is the source of this inherent feature of the ρ_{KG}: the equation (1.19) is of the second order in time and thus a time derivative emerges necessarily in (1.21). Thus, it is desirable to have an equation that would be of the first order with respect to time. To ensure the relativistic covariance, it should also be of the first order in space variables (time and space coordinates are treated on an equal footing in Lorentz transformations). In any case, one has to maintain the relativistic relation between energy and momentum (1.6) (for a moment, we come back to ordinary units). For his purpose, Dirac took the square root of (1.6) by linearizing it as follows,

$$E = c\alpha^j p^j + \beta mc^2\,, \tag{1.22}$$

where α^j, $j = 1, 2, 3$, and β are some constant coefficients (summation over the index j is understood here, so $\alpha^j p^j$ can also be written as $\vec{\alpha} \cdot \vec{p}$). Now, employing the correspondence (1.2) one arrives at the equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-i\hbar c\alpha^j \nabla^j + \beta mc^2\right)\psi\,. \tag{1.23}$$

The consistency condition for such an equation is that upon squaring it, one should recover the Klein–Gordon equation (which corresponds trivially to the energy–momentum relation (1.6)). Before squaring Eq. (1.23) one must clarify a simple point: If one has an equation $A\psi = B\psi$ with A, B being some operators, it does not imply automatically that $A^2\psi = B^2\psi$. Indeed, if A and B do not commute, the latter identity is not guaranteed. However, if $AB = BA$, then obviously $A\psi = B\psi \Rightarrow A^2\psi = AB\psi = BA\psi = B^2\psi$. Eq. (1.23) clearly corresponds to the case $[A, B] = 0$, since the time derivative commutes with ∇^j on the right-hand side. One may now square Eq. (1.23) with confidence; the only caveat is that one must not assume *a priori* that the coefficients α^j, β commute (they cannot be ordinary numbers). Thus, one gets

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = \left[-\hbar^2c^2\big(\vec{\alpha}\cdot\vec{\nabla}\big)\big(\vec{\alpha}\cdot\vec{\nabla}\big) - i\hbar c\cdot mc^2\left(\vec{\alpha}\beta+\beta\vec{\alpha}\right)\cdot\vec{\nabla} + \beta^2m^2c^4\right]\psi\,. \tag{1.24}$$

In Eq. (1.24) one has

$$\big(\vec{\alpha}\cdot\vec{\nabla}\big)\big(\vec{\alpha}\cdot\vec{\nabla}\big) = \alpha^j\alpha^k\nabla^j\nabla^k = \frac{1}{2}\big\{\alpha^j,\alpha^k\big\}\nabla^j\nabla^k + \frac{1}{2}\big[\alpha^j,\alpha^k\big]\nabla^j\nabla^k\,, \tag{1.25}$$

but the last term in (1.25) vanishes, since $\nabla^j\nabla^k = \nabla^k\nabla^j$. In order to turn Eq. (1.24) into the form of the Klein–Gordon equation, the coefficients α^j must obviously satisfy the identities

$$\begin{aligned}\big\{\alpha^j,\alpha^k\big\} &= 2\delta^{jk}\,,\\ \big\{\beta,\alpha^j\big\} &= 0\,,\\ \beta^2 &= 1\,.\end{aligned} \tag{1.26}$$

So, it is clear that α^j and β must be matrices rather than ordinary numbers, as we have rightly anticipated before. Moreover, the equation (1.23) has a "Schrödinger-like" form; the operator on its right-hand side could be interpreted as a Hamiltonian that should be Hermitian (self-adjoint). It means that one should impose an additional constraint on α^j and β, namely

$$\big(\alpha^j\big)^\dagger = \alpha^j\,, \qquad \beta^\dagger = \beta\,. \tag{1.27}$$

Now the question is, what can be matrices satisfying (1.26) and (1.27). First of all, it is not difficult to show that the dimension of such matrices must be even. Indeed, (1.26) means, in particular, that

$$\alpha^j\alpha^k = -\alpha^k\alpha^j \qquad \text{for} \quad j\neq k\,, \tag{1.28}$$

and

$$\big(\alpha^j\big)^2 = 1 \qquad \text{for} \quad j = 1,2,3\,. \tag{1.29}$$

Let us now consider the determinants of the matrix products in (1.28). One has

$$\det\alpha^j\,\det\alpha^k = \det(-\mathbb{1})\,\det\alpha^k\,\det\alpha^j = (-1)^d\,\det\alpha^j\,\det\alpha^k\,, \tag{1.30}$$

where d is the dimension of matrices in question. Obviously, $\det\alpha^j \neq 0$ because of (1.29), so (1.30) implies $(-1)^d = 1$, i.e. d is even.

The simplest choice would be $d = 2$, but it does not work; the point is that there are not four mutually anticommuting 2×2 matrices. Indeed, for α^j, $j = 1,2,3$, one could take the Pauli matrices σ^j, but then there is no non-trivial β that would anticommute with them. Proving this statement independently is left to the reader as an instructive algebraic exercise.

The next try is $d = 4$ and we will see shortly that it does work. Needless to say, it then also means that the wave function ψ in Eq. (1.23) has four components. Before showing an explicit example of 4×4 matrices satisfying (1.26) and (1.27), let us mention another general property of the matrices in question. It is easy to show that matrices α^j and β are traceless,

$$\begin{aligned}\operatorname{Tr}\alpha^j &= 0\,, \qquad j = 1,2,3\,,\\ \operatorname{Tr}\beta &= 0\,.\end{aligned} \tag{1.31}$$

Let us prove e.g. the first identity (1.31). Since $\beta^2 = 1$, one may write

$$\operatorname{Tr}\alpha^j = \operatorname{Tr}\big(\beta^2\alpha^j\big) = \operatorname{Tr}\big(\beta\alpha^j\beta\big) = -\operatorname{Tr}\big(\alpha^j\beta^2\big) = -\operatorname{Tr}\alpha^j\,,$$

so that indeed $\operatorname{Tr}\alpha^j = 0$. Note that we have utilized just the trace cyclicity and the anticommutation property $\beta\alpha^j = -\alpha^j\beta$. The second identity (1.31) can be proved in the similar way, employing the same trick with e.g. $\left(\alpha^1\right)^2 = 1$.

Finally, let us display an explicit example of the 4 × 4 matrices satisfying (1.26) and (1.27). They are

$$\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \qquad j = 1, 2, 3, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{1.32}$$

where σ^j are the familiar Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.33}$$

and $\mathbb{1}$ stands for the 2 × 2 unit matrix. It is straightforward to verify that the matrices (1.32) have indeed the required properties. The representation (1.32) is used frequently in practical calculations and is called the **standard representation**.

In the next chapter we will see that the magic Dirac's trick of taking the square root of the energy–momentum relation (1.6) in terms of 4 × 4 matrix coefficients leads indeed to a successful description of the electron. The great leap from the simple kinematical relation (1.6) to the deep quantum equation with rich physical contents makes the Dirac equation one of the most remarkable achievements of the 20th century physics. Note that Dirac received the Nobel Prize in 1933 together with E. Schrödinger. Many historical details concerning the Dirac's discovery can be found in the book [9].

Chapter 2

Physical contents of Dirac equation: preliminary discussion

As we have noted in the preceding chapter, the prime motivation for finding an alternative to the Klein–Gordon equation was the requirement that the probability defined in terms of a quantum mechanical wave function should be positive. So, let us now examine this problem for the Dirac equation; for convenience, we return to the natural units. Eq. (1.23) then reads

$$i\,\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m\psi \tag{2.1}$$

(we will use the standard representation (1.32) in what follows). Let us recall that ψ is a four-component wave function that is conventionally written as a column

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}. \tag{2.2}$$

Upon Hermitian conjugation of Eq. (2.1) one has

$$-i\,\frac{\partial\psi^\dagger}{\partial t} = i\,\vec{\nabla}\psi^\dagger\vec{\alpha} + m\psi^\dagger\beta\,, \tag{2.3}$$

where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$, and we have utilized the hermiticity property (1.27) of $\vec{\alpha}$ and β. Multiplying Eq. (2.1) by $\psi^\dagger$ from the left and (2.3) by ψ from the right, and taking then the difference of the two equations, one gets immediately

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \vec{\nabla}(\psi^\dagger\vec{\alpha}\psi) = 0\,, \tag{2.4}$$

which is the anticipated continuity equation. Thus we may identify the probability density and the probability current as

$$\rho_{\text{Dirac}} = \psi^\dagger\psi\,, \qquad \vec{j}_{\text{Dirac}} = \psi^\dagger\vec{\alpha}\psi\,. \tag{2.5}$$

The positivity of the ρ_{Dirac} is obvious, since

$$\psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2\,. \tag{2.6}$$

This is an expected result, due to the fact that the Dirac equation (2.1) is, in a sense, “square root of Klein–Gordon equation”; more precisely, it is an evolution equation of the 1st order in time, having the form

$$i\,\frac{\partial\psi}{\partial t} = H\,\psi\,, \tag{2.7}$$

where H is the Dirac Hamiltonian

$$H = -i\vec{\alpha} \cdot \vec{\nabla} + \beta m \,. \tag{2.8}$$

Thus, the time evolution is generated by an operator of energy, as it should be, in accordance with the general principles of quantum theory.

A next issue is the angular momentum. Let us start with orbital angular momentum, defined in the standard way as $\vec{L} = \vec{x} \times \vec{p}$, where $\vec{p}$ is the (linear) momentum $\vec{p} = -i\vec{\nabla}$. As we know, $\vec{L}$ commutes with the non-relativistic Hamiltonian in the Schrödinger equation (1.4). For the Dirac Hamiltonian (2.8) one gets, employing the canonical commutation relation $[x^j, p^k] = i\delta^{jk}$,

$$[H, \vec{L}] = -i(\vec{\alpha} \times \vec{p}) \,. \tag{2.9}$$

Let us remark that the vector product in (2.9) is defined formally as usual, i.e.

$$(\vec{\alpha} \times \vec{p})^j = \varepsilon^{jkl} \alpha^k p^l \,.$$

So, apparently, there is something missing, since any decent angular momentum should be an integral of motion for the free particle, i.e. the corresponding operator should commute with the Hamiltonian. In other words, the fact that $[H, \vec{L}] \neq 0$ is a hint that we are on the right track towards the electron spin. A good candidate for such an additional ingredient of the full angular momentum is guessed quite easily. Let us consider the 4×4 matrices

$$\vec{S} = \frac{1}{2}\vec{\Sigma} \,, \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} , \tag{2.10}$$

and recall that the Pauli matrices have the commutation relations

$$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l \,. \tag{2.11}$$

This means that the matrices $\vec{S}$ defined by (2.10) satisfy

$$[S_j, S_k] = i\varepsilon_{jkl} S_l \,, \tag{2.12}$$

which, of course, is a set of commutation relations for components of an angular momentum. Needless to say, the matrices $\vec{S}$ possess eigenvalues $\pm 1/2$ (because $(\sigma_j)^2 = 1$ for $j = 1, 2, 3$). Now we may evaluate the commutator $[H, \vec{S}]$. Clearly, $\vec{S}$ commutes with the diagonal matrix β (see (1.32)). Concerning the commutator involving $\vec{\alpha}$, one gets first

$$[\alpha^j, \Sigma^k] = \begin{pmatrix} 0 & 2i\varepsilon^{jkl}\sigma^l \\ 2i\varepsilon^{jkl}\sigma^l & 0 \end{pmatrix} ,$$

so that

$$[H, \Sigma^k] = 2i(\vec{\alpha} \times \vec{p})^k \,. \tag{2.13}$$

Summarizing the results of our simple algebraic exercise, we have

$$\begin{aligned} [H, \vec{L}] &= -i(\vec{\alpha} \times \vec{p}) \,, \\ [H, \vec{S}] &= i(\vec{\alpha} \times \vec{p}) \,, \end{aligned} \tag{2.14}$$

and thus

$$[H, \vec{J}] = 0 \,, \tag{2.15}$$

with

$$\vec{J} = \vec{L} + \vec{S}\,. \tag{2.16}$$

Thus, in such a straightforward manner we have recovered the electron spin as a part of the conserved total angular momentum (2.16).

Let us now recall the problem of negative energy solutions of the Klein–Gordon equation, mentioned in the preceding chapter (cf. (1.12)). One may wonder whether the Dirac equation suffers an analogous difficulty. For clarifying this point, we are going to consider the solution of Eq. (2.1) in the form of a plane wave involving the usual factor $\exp\left[-i(Et - \vec{p}\cdot\vec{x})\right]$. To make our discussion as simple as possible, we will restrict ourselves to the case of a particle at rest, i.e. set $\vec{p} = 0$. Eq. (2.1) is then reduced to

$$i\,\frac{\partial\psi}{\partial t} = \beta m \psi\,. \tag{2.17}$$

Taking into account the block diagonal structure of the matrix β ((1.32), it is useful to split the ψ as

$$\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{2.18}$$

where φ and χ are two-component column vectors. Eq. (2.17) is then recast as

$$i\,\frac{\partial\varphi}{\partial t} = m\varphi\,, \tag{2.19}$$

$$i\,\frac{\partial\chi}{\partial t} = -m\chi\,. \tag{2.20}$$

Thus, two linearly independent solutions of Eq. (2.19) may be written e.g. as

$$\varphi_{(1)} = e^{-imt}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \varphi_{(2)} = e^{-imt}\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.21}$$

and similarly for (2.20),

$$\chi_{(1)} = e^{imt}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_{(2)} = e^{imt}\begin{pmatrix}0\\1\end{pmatrix}. \tag{2.22}$$

In this way, we obtain a set of four independent solutions of Eq. (2.1)

$$\psi_{(1)} = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi_{(2)} = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \psi_{(3)} = e^{imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi_{(4)} = e^{imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{2.23}$$

Obviously, $\psi_{(1)}$ and $\psi_{(2)}$ correspond to the positive rest energy $E = m$, while $\psi_{(3)}$ and $\psi_{(4)}$ carry negative energy $E = -m$ (they are also characterized by the two possible spin projections to the third axis, up and down $(\pm 1/2)$). It is interesting to notice that in the considered case, the existence of the negative energy solutions is a consequence of the specific structure of the matrix β. If β were 4×4 unit matrix, we would have only a solution with positive energy. But, alas, β can never be the unit matrix because of the required anticommutation relations (1.26). As we have already noted in the preceding chapter, the appearance of negative energy solutions is a generic feature of the equations of relativistic quantum mechanics. We will discuss the plane-wave solutions of Dirac equation in detail later on.

The last topic that we are going to discuss here is a derivation of the spin magnetic moment of the electron. Soon after the birth of relativistic quantum mechanics this was indeed one of the most remarkable achievements of the Dirac theory, so it certainly deserves a detailed exposition.

To this end, one has to start with the Dirac equation for the electron in an external electromagnetic field. Using the scalar potential ϕ and vector potential $\vec{A}$, one may write the relevant equation as

$$i\,\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\cdot(-i\vec{\nabla} - e\vec{A}) + e\phi + \beta m\right]\psi\,. \tag{2.24}$$

Note that the form (2.24) represents the so-called **minimal electromagnetic interaction** and satisfies certainly the requirement of gauge invariance (invariance under gauge transformations of the potentials ϕ and $\vec{A}$). In fact, it is not the most general choice, but coincides with the recipe to be employed later on, in quantum electrodynamics. More comments on a possible extension of the gauge invariant electromagnetic interaction within the framework of Dirac equation are deferred to the Chapter 13.

Our ultimate goal is to get the non-relativistic two-component **Pauli equation**, from which one can extract easily the value of the magnetic moment in question. For this purpose, we will separate upper and lower components of the wave function ψ as

$$\psi = \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\chi}\end{pmatrix}. \tag{2.25}$$

Then, denoting

$$-i\vec{\nabla} - e\vec{A} = \vec{\pi}\,, \tag{2.26}$$

Eq. (2.24) is recast as a pair of coupled two-component equations

$$\begin{aligned} i\,\frac{\partial\widetilde{\varphi}}{\partial t} &= (\vec{\sigma}\cdot\vec{\pi})\widetilde{\chi} + (e\phi + m)\widetilde{\varphi}\,,\\ i\,\frac{\partial\widetilde{\chi}}{\partial t} &= (\vec{\sigma}\cdot\vec{\pi})\widetilde{\varphi} + (e\phi - m)\widetilde{\chi}\,. \end{aligned} \tag{2.27}$$

Throughout our calculation we will have in mind a situation close to the non-relativistic limit; thus, it is convenient to factorize in the wave function a part corresponding to the rest energy. (cf. (2.21)), i.e. introduce the Ansatz

$$\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\chi}\end{pmatrix} = e^{-imt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{2.28}$$

Inserting (2.28) into Eq. (2.27) one gets, after a simple manipulation,

$$i\,\frac{\partial\varphi}{\partial t} = (\vec{\sigma}\cdot\vec{\pi})\chi + e\phi\varphi\,, \tag{2.29a}$$

$$i\,\frac{\partial\chi}{\partial t} = (\vec{\sigma}\cdot\vec{\pi})\varphi + e\phi\chi - 2m\chi\,. \tag{2.29b}$$

We consider weak fields, in particular $e\phi \ll m$, as well as a small kinetic energy; the latter assumption may be expressed, technically, as

$$\frac{\partial\chi}{\partial t} \ll m\chi\,.$$

Thus, in Eq. (2.29b) we will neglect $\partial\chi/\partial t$ and $e\phi\chi$ in comparison with $2m\chi$. Consequently, the function χ can be approximately written as

$$\chi \doteq \frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})\varphi\,. \tag{2.30}$$

Using the last expression in Eq. (2.29a), we have

$$i\frac{\partial\varphi}{\partial t}=\frac{1}{2m}(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})\varphi+e\phi\varphi\,. \tag{2.31}$$

To work out the right-hand side of Eq. (2.31), one may utilize the familiar identity for Pauli matrices

$$\sigma_j\sigma_k=\delta_{jk}\cdot\mathbb{1}+i\varepsilon_{jkl}\sigma_l\,. \tag{2.32}$$

From (2.32) one then gets

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})=\vec{\pi}^2+i\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi})\,. \tag{2.33}$$

One must treat the vector product carefully, since $\vec{\pi}$ is a differential operator. So, one has to evaluate it by letting it act on an arbitrary test function f; one obtains, after some manipulations,

$$(\vec{\pi}\times\vec{\pi})^j f=ie(\vec{\nabla}\times\vec{A})^j f\,,$$

so that

$$\vec{\pi}\times\vec{\pi}=ie(\vec{\nabla}\times\vec{A})=ie\vec{B}\,, \tag{2.34}$$

where $\vec{B}$ is the magnetic field (the reader is encouraged to reproduce independently the result (2.34)). In total, we thus have

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})=(-i\vec{\nabla}-e\vec{A})^2-e\vec{\sigma}\cdot\vec{B}\,.$$

The two-component equation (2.31) thus becomes

$$i\frac{\partial\varphi}{\partial t}=\left[\frac{1}{2m}(\vec{p}-e\vec{A})^2+e\phi-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}\right]\varphi\,, \tag{2.35}$$

and this is the anticipated Pauli equation. Obviously, the last term in the square brackets represents an interaction of magnetic moment with magnetic field $\vec{B}$. Since the Pauli matrices have eigenvalues ± 1, one may conclude that the value of the magnetic moment in question is $e/(2m)$ (i.e. one **Bohr magneton**). Note that Wolfgang Pauli formulated Eq. (2.35) in 1927 as a phenomenological description of the electron moving in an external field; he then used the empirically known value of the spin magnetic moment. The derivation described above is actually a **prediction** of the relevant value, made on the basis of a more fundamental equation (though restricted to the minimal electromagnetic interaction). This is why the result (2.35) obtained as a non-relativistic approximation of Dirac equation is extolled as a true achievement.

One more remark is in order here. Magnetic moment of a particle is usually characterized also by its **gyromagnetic ratio**, which is the ratio of the magnetic moment to the angular momentum. It is a well-known fact that for the orbital motion, the gyromagnetic ratio is equal to $e/(2m)$ (this holds both in classical and in quantum theory). For the spin magnetic moment we obviously have the gyromagnetic ratio e/m, since the magnitude of the spin projection is $1/2$. Thus, the spin magnetic moment of electron does not obey the "normal" rule and differs from it by a dimensionless factor called simply ***g*-factor**, here equal to 2. The g-factor has become a usual way of description of intrinsic magnetic moment of subatomic particles.

The above-described elegant derivation of the electron spin magnetic moment, in particular the natural explanation of the "anomalous" value $g=2$ for the g-factor was certainly a great success of the Dirac theory in 1928. In fact, even more remarkable was the continuation of this success story some 20 years later. It turned out that quantum electrodynamics (QED) leads to a tiny correction to the Dirac's prediction. The correction is of relative order of one per-mille; it was found experimentally in 1947 and subsequently calculated theoretically by Julian Schwinger, one of the founding fathers of modern QED. This achievement corroborated strongly the QED as the relevant model of quantum field theory capable to describe the most subtle electromagnetic phenomena. We will discuss this topic in detail in Chapter 50.

Chapter 3

Covariant form of Dirac equation. Fun with γ-matrices

The main topic of this and the following chapter is the relativistic invariance of the Dirac equation. Before proceeding to this extensive theme, let us return briefly to the Klein–Gordon equation. In that case, the relativistic invariance is almost obvious, since the d'Alembert operator $\Box = \partial^\mu \partial_\mu$ has the structure of a scalar product in the four-dimensional spacetime. So, when passing from one Lorentz reference frame to another, with coordinates transformed as $x' = \Lambda x$ (with Λ being the matrix of a Lorentz transformation), one can get along with a trivial transformation of the wave function

$$\psi'(x') = \psi(x)\,. \tag{3.1}$$

As we will see in the next chapter, in the case of Dirac equation the situation is much more interesting, i.e. far from trivial. For a proper assessment of this problem, it is convenient to recast first the Dirac equation in a form that is more symmetric with respect to spacetime coordinates; this is what is meant by the term "covariant form" in the title of this chapter.

For a moment, let us use the ordinary units with $\hbar \neq 1$, $c \neq 1$. As we know, Dirac equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c\, \vec{\alpha} \cdot \vec{\nabla} \psi + \beta m c^2 \psi\,. \tag{3.2}$$

Introducing the usual notation $x_0 = ct$ and taking into account that ∇^j is defined, conventionally, as $\partial / \partial x^j = \partial_j$, one may rewrite Eq. (3.2) as

$$i\hbar c \frac{\partial \psi}{\partial x_0} = -i\hbar c\, \vec{\alpha} \cdot \vec{\nabla} \psi + m c^2 \beta \psi\,, \tag{3.3}$$

and this subsequently becomes

$$i\hbar \beta \partial_0 \psi + i\hbar \beta \alpha^j \partial_j \psi - mc\psi = 0\,. \tag{3.4}$$

If we now denote

$$\gamma^0 = \beta\,, \qquad \gamma^j = \beta \alpha^j\,, \tag{3.5}$$

then Eq. (3.4) can be rewritten as

$$i\gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar} \psi = 0\,. \tag{3.6}$$

One may notice the appearance of the inverse Compton length in the second term; this, of course, was to be expected on dimensional grounds. Thus, in natural units, Eq. (3.6) reads

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0\,, \tag{3.7}$$

and this is the promised "covariant form" of Dirac equation, which will be our staple food from now on. The Dirac matrices γ^μ will be called simply gamma matrices, or γ-matrices in what follows. Notice that Eq. (3.7) seems to look covariant, since the term $\gamma^\mu \partial_\mu$ has, at first sight, the form of a scalar product in Minkowski spacetime; however, γ^μ, $\mu = 0, 1, 2, 3$, are fixed 4×4 matrices to be used in any reference frame, so one should not jump to conclusions at this point. Anyway, a most economical form of Eq. (3.7), utilizing the scalar product symbol, is perhaps

$$i\gamma \cdot \partial\psi = m\psi \,, \tag{3.8}$$

and this is precisely what is engraved in the commemorative marker in Dirac's honour in Westminster Abbey (it is there since 1995).

So, from now on, we will work with the set of matrices γ^μ introduced in (3.5); it is also convenient to employ formally the rule for raising and lowering the Lorentz indices and define $\gamma_\mu = g_{\mu\nu}\gamma^\nu$, i.e.

$$\gamma_0 = \gamma^0 \,, \qquad \gamma_j = -\gamma^j \,. \tag{3.9}$$

To begin with, we should rewrite the anticommutation relations (1.26) in terms of γ^μ. This is an easy exercise; we have

$$\begin{aligned} \{\gamma^0, \gamma^j\} &= \{\beta, \beta\alpha^j\} = \beta^2\alpha^j + \beta\alpha^j\beta = 0 \,, \\ \{\gamma^j, \gamma^k\} &= \{\beta\alpha^j, \beta\alpha^k\} = \beta\alpha^j\beta\alpha^k + \beta\alpha^k\beta\alpha^j = -\{\alpha^j, \alpha^k\} = -2\delta^{jk} \cdot \mathbb{1} \,, \end{aligned}$$

and, of course,

$$\{\gamma^0, \gamma^0\} = 2\beta^2 = 2 \cdot \mathbb{1} \,,$$

where $\mathbb{1}$ denotes the 4×4 unit matrix. Thus, we can summarize the above results as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1} \tag{3.10}$$

(for brevity, we will usually omit $\mathbb{1}$ when using (3.10)). In mathematics, the anticommutation relations (3.10) are known to correspond to generators of the so-called **Clifford algebra**. Note that (3.10) means, in particular,

$$(\gamma^0)^2 = \mathbb{1} \,, \qquad (\gamma^j)^2 = -\mathbb{1} \,. \tag{3.11}$$

Further, let us see what becomes of the hermiticity relations (1.27). Obviously, one has

$$(\gamma^0)^\dagger = \gamma^0 \,, \qquad (\gamma^j)^\dagger = (\beta\alpha^j)^\dagger = \alpha^j\beta = -\gamma^j \,. \tag{3.12}$$

So, taking into account (3.10), (3.12) may be summarized as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \,. \tag{3.13}$$

This last relation is one of the identities that will be used very frequently in our forthcoming calculations.

It is highly useful to introduce a fifth γ-matrix, denoted traditionally as γ_5, which is proportional to the product $\gamma_0\gamma_1\gamma_2\gamma_3$. The salient feature of such a matrix product is that it anticommutes with any γ^μ, $\mu = 0, 1, 2, 3$ (the reader is encouraged to check this statement independently). Note that we fix the definition of γ_5 conventionally as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \,. \tag{3.14}$$

So, we have

$$\{\gamma_5, \gamma^\mu\} = 0\,, \qquad \mu = 0, 1, 2, 3\,. \tag{3.15}$$

Other basic properties of the γ_5 shown in (3.14) are

$$(\gamma_5)^2 = \mathbb{1}\,, \qquad (\gamma_5)^\dagger = \gamma_5 \tag{3.16}$$

(again, proving (3.16) is left to the reader as an easy exercise).

The rest of this chapter is devoted to a rather detailed discussion of various properties of the gamma matrices; it is just an algebra, no physics. So, this is the would-be "fun" mentioned in the heading (though, admittedly, the "fun" in the present context is a matter of personal taste — obviously, the trick here was to lure the reader into studying an otherwise somewhat boring subject).

First of all, it is useful to master some simple formulae for traces of products of γ-matrices. We already know (see (1.31)) that traces of the original Dirac matrices α^j and β are zero; this finding is easily reproduced for the γ-matrices as well. With the matrix γ_5 at hand, it is straightforward to prove that the trace of a product of odd number of γ-matrices is zero; symbolically,

$$\mathrm{Tr}(\text{odd}\,\#) = 0\,. \tag{3.17}$$

The proof can be left to the reader as an instructive exercise. Hint: Use $\gamma_5^2 = \mathbb{1}$, the anticommutation property (3.15) and trace cyclicity. By the way, using a similar trick, one can prove as well that

$$\mathrm{Tr}\,\gamma_5 = 0 \tag{3.18}$$

(to this end, one may use e.g. $\gamma_0^2 = \mathbb{1}$).

For products of an even number $n = 2k$ of γ-matrices one gets a series of formulae that have quite uniform structure. Let's start with $n = 2$. One has certainly $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = \mathrm{Tr}(\gamma_\nu\gamma_\mu)$, so that, employing (3.10),

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu) = \frac{1}{2}\,\mathrm{Tr}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \frac{1}{2}\cdot 2g_{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} = 4g_{\mu\nu}\,. \tag{3.19}$$

How about $n = 4$? We have, using (3.10),

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \mathrm{Tr}\big[(2g_{\mu\nu} - \gamma_\nu\gamma_\mu)\gamma_\rho\gamma_\sigma\big] = 2g_{\mu\nu}\,\mathrm{Tr}(\gamma_\rho\gamma_\sigma) - \mathrm{Tr}(\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma)\,. \tag{3.20}$$

Now, we can go on anticommuting the γ_μ with γ_ρ and then with γ_σ. In this way, we end up with

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 2g_{\mu\nu}\,\mathrm{Tr}(\gamma_\rho\gamma_\sigma) - 2g_{\mu\rho}\,\mathrm{Tr}(\gamma_\nu\gamma_\sigma) + 2g_{\mu\sigma}\,\mathrm{Tr}(\gamma_\nu\gamma_\rho) - \mathrm{Tr}(\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu)\,. \tag{3.21}$$

However, in the last term on the right-hand side of (3.21) we can use the trace cyclicity and one thus gets, eventually,

$$\begin{aligned}\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= \frac{1}{2}\big[2g_{\mu\nu}\,\mathrm{Tr}(\gamma_\rho\gamma_\sigma) - 2g_{\mu\rho}\,\mathrm{Tr}(\gamma_\nu\gamma_\sigma) + 2g_{\mu\sigma}\,\mathrm{Tr}(\gamma_\nu\gamma_\rho)\big] \\ &= 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})\,, \end{aligned} \tag{3.22}$$

where we have utilized the preceding result (3.19).

The above example makes it clear how to proceed further, i.e. for $n \geq 6$: one moves the first γ-matrix in the product step by step (employing the basic anticommutation relation (3.10)) to the last position, and then the trace cyclicity can be used. On the way, one encounters products with the number of γ-matrices less by two, so one can utilize the result for the preceding member of the whole hierarchy. Thus, it is quite clear that the results for traces in question are expressed as products of pertinent components of the metric tensor g; for $n = 2k$ these products consist of just k factors. The resulting number N of terms for the trace with $n = 2k$ grows rapidly with n; the recursive procedure outlined above shows clearly that $N(2k) = (2k-1)N(2k-2)$, which means that

$$N(2k) = (2k-1)!! = \frac{(2k)!}{2^k\, k!} \tag{3.23}$$

(so, for $n = 6$ one gets 15 terms, for $n = 8$ there are 105 terms, etc.). One would certainly have a lot of fun computing such a trace for $n = 14$, which amounts to 135 135 terms (sic!), but rest assured that we will always get along with smaller numbers.[3] In any case, one might wonder what is it all good for; please, don't worry and be patient, you will see that the traces of products of γ-matrices will come in handy later (in QED, in particular). Perhaps one may refer to a well-known quotation (due to A. P. Chekhov (Čechov)) saying that "If in the first act (of a drama) there is a rifle hanging on the wall, then in the last act someone must fire it." (in fact, we will "fire the rifle" much earlier than in the last act of this lecture course).

There are many other special identities for γ-matrices that will be practically useful later on (they are collected in Appendix C), but now we are going to study some of their deeper structural properties that will be needed soon. The basic point of the analysis that follows is the observation that one can find an appropriate basis in the space of 4 × 4 matrices, made of products of γ-matrices. To this end, we will consider 16 matrices, denoted for convenience as Γ_A, $A = 1, \ldots, 16$, and defined as follows. First, $\Gamma_1 = \mathbb{1}$ (this can be obtained as e.g. $(\gamma_0)^2$); next, we take

$$\gamma_\mu\,, \quad \mu = 0,1,2,3 \quad : \quad \Gamma_2\,, \Gamma_3\,, \Gamma_4\,, \Gamma_5\,, \tag{3.24}$$

and then one may form products of two, three and four γ-matrices. That's the end of the story — it is clear that products of five and more γ-matrices would not bring anything new, because of (3.11) (such expressions are reduced to products of less than five γ-matrices). Thus, let us denote

$$\gamma_0\gamma_1\,, \gamma_0\gamma_2\,, \gamma_0\gamma_3\,, \gamma_1\gamma_2\,, \gamma_1\gamma_3\,, \gamma_2\gamma_3 \quad : \quad \Gamma_6\,, \ldots, \Gamma_{11}\,. \tag{3.25}$$

Further, four independent products of three γ-matrices are equivalent to

$$\gamma_0\widetilde{\gamma}_5\,, \gamma_1\widetilde{\gamma}_5\,, \gamma_2\widetilde{\gamma}_5\,, \gamma_3\widetilde{\gamma}_5 \quad : \quad \Gamma_{12}\,, \Gamma_{13}\,, \Gamma_{14}\,, \Gamma_{15}\,, \tag{3.26}$$

where $\widetilde{\gamma}_5$ is defined as

$$\widetilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \tag{3.27}$$

(we have chosen this provisional notation instead of (3.14) for simplicity). Finally, we set

$$\Gamma_{16} = \widetilde{\gamma}_5\,. \tag{3.28}$$

Using (3.10), it is easy to see that the square of any matrix Γ_A is either $\mathbb{1}$ or $-\mathbb{1}$. In particular, one has

$$\begin{aligned} \Gamma_A^2 &= \mathbb{1} && \text{for} \quad A = 1,2,6,7,8,12\,, \\ \Gamma_A^2 &= -\mathbb{1} && \text{for} \quad A = 3,4,5,9,10,11,13,14,15,16\,. \end{aligned} \tag{3.29}$$

So, the set of Γ_A, $A = 1, \ldots, 16$, has the right number of terms to be a good candidate for a basis in the considered 16-dimensional space of 4×4 matrices. Before showing that the Γ_A are indeed

---

[3] Note that discovering this remarkable number has been just serendipitous; obviously, an average QFT practitioner can hardly come across it in routine calculations.

linearly independent, we are going to present a few auxiliary statements (lemmas) describing some simple, but important, properties of the matrices Γ_A.

L1 (commutation & anticommutation): *Any pair Γ_A, Γ_B either commutes or anticommutes.* This can be proved easily by employing the anticommutation relations (3.10) and (3.15).

L2 (on traces): *$Tr\,\Gamma_A = 0$ for any $A > 1$.*
A part of this statement we have already proved before; in general, using (3.10) and (3.15) is sufficient.

L3 (on the rearrangement): *When multiplying all Γ_A's by a particular Γ_B from left or right, one gets again the same set, up to signs and the order.*
One may prove such a statement simply "by inspection" (in principle, one should produce a pertinent multiplication table with $16 \times 16 = 256$ entries).

The above lemmas are now sufficient for establishing the fact that such Γ_A's form a basis.

L4 (on the linear independence): *The matrices Γ_A, $A = 1, \ldots, 16$, are linearly independent.*
Proof: Suppose that

$$a_1\Gamma_1 + a_2\Gamma_2 + \ldots + a_{16}\Gamma_{16} = 0 \tag{3.30}$$

for some coefficients $a_1, \ldots, a_{16}$. For convenience, let us denote the linear combination on the left-hand side of (3.30) simply as L. Obviously, Eq. (3.30) implies, for any $A = 1, \ldots, 16$,

$$\mathrm{Tr}(\Gamma_A L) = 0\,. \tag{3.31}$$

Now, using lemmas L2 and L3, along with the identities (3.28), (3.29), one gets $\mathrm{Tr}(\Gamma_A L) = \pm 4a_A$, depending on whether Γ_A^2 is $\mathbb{1}$ or $-\mathbb{1}$. In any case, Eq. (3.31) thus implies $a_A = 0$ for any $A = 1, \ldots, 16$ and the statement L4 is thereby proved.

Now we are in a position to prove the following important statement:

L5: *Let M be a matrix 4×4 that commutes with any γ_μ, $\mu = 0, 1, 2, 3$. Then M is a multiple of the unit matrix.*
Proof: According to the preceding lemma L4, the matrices $\Gamma_1, \ldots, \Gamma_{16}$ form a basis; thus, the matrix M can be expressed as a linear combination

$$M = a_1\Gamma_1 + \ldots + a_{16}\Gamma_{16}\,. \tag{3.32}$$

The premise represents four conditions, namely $[M, \gamma_\mu] = 0$ for $\mu = 0, 1, 2, 3$. Let us start with $\mu = 0$. Using (3.32), the condition $[M, \gamma_0] = 0$ means

$$a_1\Gamma_1\gamma_0 + \ldots + a_{16}\Gamma_{16}\gamma_0 = a_1\gamma_0\Gamma_1 + \ldots + a_{16}\gamma_0\Gamma_{16}\,. \tag{3.33}$$

It is easy to find out that γ_0 commutes with Γ_A for $A = 1, 2, 9, 10, 11, 13, 14, 15$ and anticommutes with Γ_A for $A = 3, 4, 5, 6, 7, 8, 12, 16$. Thus, the terms involving the commuting Γ_A's drop out of Eq. (3.33), while the anticommuting Γ_A's survive, and Eq. (3.33) eventually becomes

$$a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5 + a_6\Gamma_6 + a_7\Gamma_7 + a_8\Gamma_8 + a_{12}\Gamma_{12} + a_{16}\Gamma_{16} = 0\,.$$

Of course, according to the lemma L4 this amounts to

$$a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_{12} = a_{16} = 0\,. \tag{3.34}$$

Consequently, after this first step, the expansion (3.32) is reduced to

$$M = a_1\Gamma_1 + a_2\Gamma_2 + a_9\Gamma_9 + a_{10}\Gamma_{10} + a_{11}\Gamma_{11} + a_{13}\Gamma_{13} + a_{14}\Gamma_{14} + a_{15}\Gamma_{15}\,. \tag{3.35}$$

One may now continue in this way, using for (3.35) the condition $[M, \gamma_1] = 0$. It reduces further the form of M, and then one goes on along the same line with γ_2 and γ_3. It turns out that eventually one is left with $M = a_1\Gamma_1 = a_1 \cdot \mathbb{1}$ (since Γ_1 commutes with anything), and this is precisely what we wanted to prove. The reader is urged to check independently the steps involving the commutators $[M, \gamma_\mu]$ for $\mu = 1, 2, 3$.

The above series of statements concerning matrices Γ_A culminates in a profound theorem, usually called the "fundamental theorem on γ-matrices". It can be formulated as follows.

Theorem: *Let γ^μ and γ'^μ, $\mu = 0, 1, 2, 3$, be two sets of* 4×4 *matrices satisfying the relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$. Then there exists a non-singular matrix S, unique up to a multiplicative factor, such that*

$$\gamma'^\mu = S\gamma^\mu S^{-1} \tag{3.36}$$

for any $\mu = 0, 1, 2, 3$. Further, if the γ-matrices satisfy the hermiticity conditions (3.12)*, the matrix S can be chosen to be unitary.*

The proof of this remarkable statement is somewhat long and thus we will refrain from presenting it here. The interested reader can find the proof e.g. in the book [3].

Obviously, the importance of this theorem consists in the observation that all possible realizations of the Dirac γ-matrices are equivalent. Nevertheless, some particular representations may be more convenient than others in practical calculations. So, one might use a paraphrase of the familiar sentence from a famous book by George Orwell, namely: "All representations of γ-matrices are equal, but some of them are more equal than others".

We already know at least one explicit realization of γ-matrices; more precisely, we know the so-called standard representation of $\vec{\alpha}$ and β (see (1.32)), and using (3.5) one then has

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3\,, \tag{3.37}$$

as the standard representation for γ^μ. We have noticed before that within such a representation, the non-relativistic limit is characterized by a suppression of the lower two components of the wave function with respect to the upper ones. There are at least two more examples of γ-matrix representations that are worth mentioning here. One of them is the so-called **spinor** (or **chiral**) representation (the origin of these names will become clear later), which is

$$\gamma_S^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_S^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3\,. \tag{3.38}$$

Further, there is a remarkable representation, in which all γ-matrices are purely imaginary (this in turn means that in the Dirac equation only real coefficients are then involved). It is called the **Majorana representation** and the corresponding matrices γ_M^μ can be expressed with the help of the standard γ-matrices as

$$\gamma_M^0 = \gamma^0\gamma^2\,, \quad \gamma_M^1 = -\gamma^1\gamma^2\,, \quad \gamma_M^2 = -\gamma^2\,, \quad \gamma_M^3 = \gamma^2\gamma^3\,. \tag{3.39}$$

One might also wonder whether there could be a representation involving purely real γ-matrices. The answer is no. The proof is quite tedious, but it could be a real challenge for a hard-working student. In any case, an enjoyable exercise would be finding the transformation matrices implementing the passage from the standard representation to the other two mentioned above.

After all those preparatory steps, we are ready to take up seriously the problem of relativistic invariance of the Dirac equation. This will be the main theme of the next chapter.

Chapter 4

Relativistic covariance of Dirac equation

The problem indicated in the title of this chapter can be formulated as follows. Let us consider the Dirac equation

$$i\gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} - m\,\psi(x) = 0 \tag{4.1}$$

in some coordinate frame, and a Lorentz transformation to another frame,

$$x'^{\,\mu} = \Lambda^\mu{}_\nu\, x^\nu\,. \tag{4.2}$$

The question is whether there is an appropriate linear transformation of the wave function, $\psi(x) \to \psi'(x')$, such that $\psi'(x')$ satisfies the equation

$$i\gamma^\mu \frac{\partial\psi'(x')}{\partial x'^{\,\mu}} - m\,\psi'(x') = 0\,. \tag{4.3}$$

So, suppose that

$$\psi'(x') = S\,\psi(x)\,, \tag{4.4}$$

where S is a 4×4 constant invertible matrix depending on Λ, i.e. $S = S(\Lambda)$. Our goal is to find an appropriate S corresponding to a given Λ. To this end, one may use (4.4) to express $\psi(x)$ as

$$\psi(x) = S^{-1}\psi'(x')\,. \tag{4.5}$$

Inserting now (4.5) into (4.1), one has

$$i\gamma^\mu S^{-1}\, \frac{\partial\psi'(x')}{\partial x'^{\,\lambda}} \frac{\partial x'^{\,\lambda}}{\partial x^\mu} - m\, S^{-1}\psi'(x') = 0\,. \tag{4.6}$$

However, from (4.2) it is clear that

$$\frac{\partial x'^{\,\lambda}}{\partial x^\mu} = \Lambda^\lambda{}_\mu\,. \tag{4.7}$$

Thus, using (4.7) and multiplying Eq. (4.6) by S from the left, one gets

$$iS\gamma^\mu S^{-1}\Lambda^\lambda{}_\mu\, \frac{\partial\psi'(x')}{\partial x'^{\,\lambda}} - m\psi'(x') = 0\,. \tag{4.8}$$

Obviously, if one wants to arrive at Eq. (4.3), the condition

$$\Lambda^\lambda{}_\mu S\gamma^\mu S^{-1} = \gamma^\lambda$$

is to be imposed. This can be recast in a more elegant form

$$\Lambda^\mu{}_\nu \gamma^\nu = S^{-1}\gamma^\mu S\,. \tag{4.9}$$

For a practical evaluation of $S = S(\Lambda)$, the exponential form of Λ (see the formula (A.17) in Appendix A)

$$\Lambda = \exp\left(-\frac{i}{2}\omega^{\alpha\beta} I_{\alpha\beta}\right) \tag{4.10}$$

is instrumental. Let us recall that $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ are the six independent parameters of a general continuous Lorentz transformation and $I_{\alpha\beta} = -I_{\beta\alpha}$ are the corresponding generators. It is reasonable to write S in analogy with (4.10) as

$$S = \exp\left(-\frac{i}{4}\omega^{\alpha\beta}\sigma_{\alpha\beta}\right), \tag{4.11}$$

where $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$ is a set of unknown would-be generators (the factor $1/4$ in the exponent is introduced for later convenience). In this way, the solution of the problem is reduced to finding the set of the matrices $\sigma_{\alpha\beta}$. This means that it is sufficient to consider infinitesimal transformations. Let us denote the infinitesimal parameters in(4.10) and (4.11) as $\Delta\omega^{\alpha\beta}$. We know (see (A.19)) that the form of the generators in (4.10) then leads to

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \Delta\omega^{\mu}{}_{\nu}\,, \tag{4.12}$$

or, equivalently,

$$\Lambda^{\mu\nu} = g^{\mu\nu} + \Delta\omega^{\mu\nu}\,.$$

For infinitesimal S and S^{-1} one may write

$$\begin{aligned} S &= \mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu}\,, \\ S^{-1} &= \mathbb{1} + \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu}\,. \end{aligned} \tag{4.13}$$

Using (4.12) and (4.13) in the condition (4.9) one has

$$\left(\mathbb{1} + \frac{i}{4}\sigma_{\alpha\beta}\,\Delta\omega^{\alpha\beta}\right)\gamma^{\mu}\left(\mathbb{1} - \frac{i}{4}\sigma_{\alpha\beta}\,\Delta\omega^{\alpha\beta}\right) = (g^{\mu\nu} + \Delta\omega^{\mu\nu})\gamma_{\nu}\,. \tag{4.14}$$

After some simple manipulations, (4.14) is recast as

$$-\frac{i}{4}\,\Delta\omega^{\alpha\beta}(\gamma^{\mu}\sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^{\mu}) = g^{\mu}{}_{\alpha}\,\Delta\omega^{\alpha\beta}\gamma_{\beta}\,. \tag{4.15}$$

Utilizing the antisymmetry of parameters $\Delta\omega^{\alpha\beta}$, one gets eventually the condition for the generators $\sigma_{\alpha\beta}$:

$$[\gamma_{\mu}, \sigma_{\alpha\beta}] = 2i(g_{\mu\alpha}\gamma_{\beta} - g_{\mu\beta}\gamma_{\alpha})\,. \tag{4.16}$$

For any fixed pair of indices α, β we thus have 64 equations for 16 unknowns (elements of the 4×4 matrix $\sigma_{\alpha\beta}$). So, at first sight one could say that $\sigma_{\alpha\beta}$ is overconstrained by the conditions (4.16). An uninspired way of solving Eq. (4.16) would consist in writing $\sigma_{\alpha\beta}$ as a linear combination of matrices Γ_A, $A = 1, \ldots, 16$, from the preceding chapter, and employ the commutation and anticommutation relations to fix the values of the relevant coefficients. This is possible, but rather tedious; one might call it a “poor man’s way”. Instead, let us try to guess some hint that would help us find a short cut to the desired solution. To this end, we are going to start e.g. with σ_{01}. The conditions (4.16) then give, for $\mu = 0, 1, 2, 3$,

$$\begin{aligned} [\gamma_0, \sigma_{01}] &= 2i\gamma_1\,, \\ [\gamma_1, \sigma_{01}] &= 2i\gamma_0\,, \\ [\gamma_2, \sigma_{01}] &= 0\,, \\ [\gamma_3, \sigma_{01}] &= 0\,. \end{aligned} \tag{4.17}$$

Contemplating (4.17) one may guess that σ_{01} must be proportional to $\gamma_0\gamma_1$; more precisely, a solution of (4.17) is, obviously,

$$\sigma_{01} = i\gamma_0\gamma_1 \,. \tag{4.18}$$

This is just the hint we need. One may, tentatively, generalize (4.18) to $\sigma_{\alpha\beta} = i\gamma_\alpha\gamma_\beta$ (of course, such an Ansatz is meaningful just for $\alpha \neq \beta$; for $\alpha = \beta$ the matrix $\sigma_{\alpha\beta}$ is trivial). Since we know that $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$, this is equivalent to

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] \,. \tag{4.19}$$

It is not difficult to verify that the expression (4.19) satisfies indeed Eq. (4.16). For this purpose, one may employ the elementary algebraic identity

$$[A, BC] = \{A, B\}C - B\{A, C\} \,. \tag{4.20}$$

Then

$$[\gamma_\mu, \sigma_{\alpha\beta}] = [\gamma_\mu, i\gamma_\alpha\gamma_\beta] = 2ig_{\mu\alpha}\gamma_\beta - 2ig_{\mu\beta}\gamma_\alpha \,,$$

and (4.16) is thereby proved.

So, we have guessed a particular solution of the conditions (4.16), but the question remains whether it is unique or not. To clarify this point, suppose there is another solution, denoted as $\sigma'_{\alpha\beta}$. Using (4.16) for $\sigma_{\alpha\beta}$ and $\sigma'_{\alpha\beta}$, one sees immediately that

$$[\sigma_{\alpha\beta} - \sigma'_{\alpha\beta}, \gamma_\mu] = 0$$

for any $\mu = 0, 1, 2, 3$. Then, according to the lemma L5 from preceding chapter, it must hold

$$\sigma'_{\alpha\beta} - \sigma_{\alpha\beta} = a \cdot \mathbb{1} \,,$$

where a is an arbitrary coefficient. Thus, (4.19) is in fact the general solution of Eq. (4.16), up to a multiple of unit matrix. Needles to say, such a trivial ambiguity has been clear from the very beginning; the non-trivial point here is that it is *the only* possible ambiguity. From now on, we will use (4.19) as the relevant formula for the generators $\sigma_{\alpha\beta}$.

An important remark is in order here. Our construction of the transformation matrix S in (4.9) has been based on generators $\sigma_{\alpha\beta}$ that correspond to the Lorentz generators $I_{\alpha\beta}$. More precisely, the correspondence between (4.10) and (4.11) is

$$\frac{1}{2}\sigma_{\alpha\beta} \longleftrightarrow I_{\alpha\beta} \,. \tag{4.21}$$

As we know, the generators $I_{\alpha\beta}$ satisfy commutation relations characteristic of a Lie algebra. In particular (see (A.22)),

$$[I_{\mu\nu}, I_{\rho\sigma}] = i(g_{\mu\sigma}I_{\nu\rho} + g_{\nu\rho}I_{\mu\sigma} - g_{\mu\rho}I_{\nu\sigma} - g_{\nu\sigma}I_{\mu\rho}) \,. \tag{4.22}$$

One might suspect that the matrices $\frac{1}{2}\sigma_{\alpha\beta}$ satisfy the same commutation relation (in mathematical language, it would mean that the six matrices $\frac{1}{2}\sigma_{\alpha\beta}$ constitute a representation of the Lie algebra of the Lorentz group). It turns out that it is indeed the case. For the evaluation of the relevant commutators one may use the identity (an extension of (4.20))

$$[AB, CD] = A\{B, C\}D - AC\{B, D\} - C\{A, D\}B + \{A, C\}DB \,. \tag{4.23}$$

Then, one has

$$
\begin{aligned}
[\frac{1}{2}\sigma_{\mu\nu},\frac{1}{2}\sigma_{\rho\sigma}] &= \frac{i}{2}\cdot\frac{i}{2}[\gamma_\mu\gamma_\nu,\gamma_\rho\gamma_\sigma] \\
&= -\frac{1}{4}(2g_{\nu\rho}\gamma_\mu\gamma_\sigma - 2g_{\nu\sigma}\gamma_\mu\gamma_\rho - 2g_{\mu\sigma}\gamma_\rho\gamma_\nu + 2g_{\mu\rho}\gamma_\sigma\gamma_\nu) \\
&= -\frac{1}{2}\Bigg[-g_{\nu\sigma}\left(\frac{1}{2}\{\gamma_\mu,\gamma_\rho\} + \frac{1}{2}[\gamma_\mu,\gamma_\rho]\right) + g_{\nu\rho}\left(\frac{1}{2}\{\gamma_\mu,\gamma_\sigma\} + \frac{1}{2}[\gamma_\mu,\gamma_\sigma]\right) \\
&\qquad - g_{\mu\sigma}\left(\frac{1}{2}\{\gamma_\rho,\gamma_\nu\} + \frac{1}{2}[\gamma_\rho,\gamma_\nu]\right) + g_{\mu\rho}\left(\frac{1}{2}\{\gamma_\sigma,\gamma_\nu\} + \frac{1}{2}[\gamma_\sigma,\gamma_\nu]\right)\Bigg] \\
&= -\frac{1}{2}(\cancel{g_{\nu\rho}g_{\mu\sigma}} + g_{\nu\rho}\frac{1}{i}\sigma_{\mu\sigma} - \cancel{g_{\nu\sigma}g_{\mu\rho}} - g_{\nu\sigma}\frac{1}{i}\sigma_{\mu\rho} \\
&\qquad - \cancel{g_{\mu\sigma}g_{\rho\nu}} - g_{\mu\sigma}\frac{1}{i}\sigma_{\rho\nu} + \cancel{g_{\mu\rho}g_{\sigma\nu}} + g_{\mu\rho}\frac{1}{i}\sigma_{\sigma\nu}) \\
&= i\left(g_{\mu\sigma}\frac{1}{2}\sigma_{\nu\rho} + g_{\nu\rho}\frac{1}{2}\sigma_{\mu\sigma} - g_{\mu\rho}\frac{1}{2}\sigma_{\nu\sigma} - g_{\nu\sigma}\frac{1}{2}\sigma_{\mu\rho}\right),
\end{aligned}
\tag{4.24}
$$

and this is precisely the anticipated commutation relation for the generators $\sigma_{\alpha\beta}$ that matches Eq. (4.22).

Thus, in our straightforward way we have discovered a four-dimensional representation of the Lorentz group (or Lorentz algebra, if you want). It is rightly called the **bispinor** (or **Dirac spinor**) as we will see shortly. An elementary mathematical theory of representations of Lorentz group is described briefly in Appendix B.

It will be certainly instructive to present some explicit examples of the matrix S implementing the transformation (4.4) of Dirac wave function. First, let us consider a spatial rotation around the third axis of coordinate system, i.e. in the plane (12). So, the relevant transformation (4.2) is given by

$$
\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{4.25}
$$

The corresponding infinitesimal form thus amounts to

$$
\begin{aligned}
x^{1'} &= x^1 + \delta\varphi\, x^2\,, \\
x^{2'} &= -\delta\varphi\, x^1 + x^2\,.
\end{aligned}
\tag{4.26}
$$

Using the notation (4.12), this means

$$
(\Delta\omega)^1{}_2 = \delta\varphi\,, \qquad (\Delta\omega)^2{}_1 = -\delta\varphi\,. \tag{4.27}
$$

It is easy to see that raising the indices in (4.27) leads to

$$
(\Delta\omega)^{12} = -\delta\varphi\,, \qquad (\Delta\omega)^{21} = \delta\varphi\,. \tag{4.28}
$$

Then, according to the formula (4.13), one has

$$
S = \mathbb{1} - 2\cdot\frac{i}{4}(\Delta\omega)^{12}\sigma_{12} = \mathbb{1} + \frac{i}{2}\delta\varphi\sigma_{12} \tag{4.29}
$$

for the infinitesimal transformation, i.e. for the representation of the finite rotation (4.25) one may write

$$
S(\varphi) = \exp\left(\frac{i}{2}\varphi\sigma_{12}\right). \tag{4.30}
$$

Now, in the standard representation of γ-matrices one gets

$$\sigma_{12} = i\gamma_1\gamma_2 = i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \Sigma_3 \,.$$

Thus, the result (4.30) can be recast as

$$S(\varphi) = \exp\left(\frac{i}{2}\varphi\Sigma_3\right). \tag{4.31}$$

Expanding the exponential (4.31) in Taylor series, one gets, taking into account that $(\Sigma_3)^2 = \mathbb{1}$,

$$S(\varphi) = \cos\frac{\varphi}{2}\cdot\mathbb{1} + i\sin\frac{\varphi}{2}\Sigma_3\,. \tag{4.32}$$

This means, in particular,

$$S(2\pi) = -\mathbb{1}\,. \tag{4.33}$$

Thus, the full rotation with $\varphi = 360°$ changes the sign of a wave function in question. This is a typical property of spinors (well-known already from the non-relativistic description of a spin-1/2 particle in terms of a two-component wave function). For this reason, it is natural to call the four-component Dirac wave function the bispinor (in Appendix B, one may find a more precise explanation of this concept).

Next, let us consider the case of a Lorentz boost; in particular, the example we take up is the uniform motion along the coordinate axis 1 with a velocity v. The corresponding transformation of spacetime coordinates reads

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \operatorname{ch}\varphi & -\operatorname{sh}\varphi & 0 & 0 \\ -\operatorname{sh}\varphi & \operatorname{ch}\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{4.34}$$

where

$$\operatorname{ch}\varphi = \frac{1}{\sqrt{1-v^2}}\,, \qquad \operatorname{sh}\varphi = \frac{v}{\sqrt{1-v^2}}\,.$$

For the infinitesimal transformation we then have

$$\begin{aligned} x^{0'} &= x^0 - \delta\varphi\, x^1\,, \\ x^{1'} &= -\delta\varphi\, x^0 + x^1\,. \end{aligned} \tag{4.35}$$

Using the notation (4.12), this means

$$(\Delta\omega)^0{}_1 = -\delta\varphi\,, \qquad (\Delta\omega)^1{}_0 = -\delta\varphi\,.$$

The corresponding infinitesimal transformation of the Dirac wave function is thus, according to (4.13),

$$S = \mathbb{1} - 2\cdot\frac{i}{4}(\Delta\omega)^{01}\sigma_{01} = \mathbb{1} - \frac{i}{2}\delta\varphi\sigma_{01}\,,$$

so that the relevant finite transformation may be written as

$$S(\varphi) = \exp\left(-\frac{i}{2}\varphi\sigma_{01}\right). \tag{4.36}$$

In the standard representation of γ-matrices one gets

$$\sigma_{01} = i\gamma_0\gamma_1 = i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.$$

Obviously, $(\sigma_{01})^2 = -\mathbb{1}$, and the exponential (4.36) can thus be worked out as

$$S(\varphi) = \mathrm{ch}\frac{\varphi}{2} \cdot \mathbb{1} - i\,\mathrm{sh}\frac{\varphi}{2}\sigma_{01}\,. \tag{4.37}$$

In this way, we have established Lorentz covariance of the Dirac equation. Up to now we have considered continuous Lorentz transformations (spatial rotations and boosts) described, in general, by six parameters. The case of discrete symmetries will be discussed in the next chapter.

To extend our technical tools for "diracology", let us now mention a simple but very useful formula that relates S^{-1} and $S^\dagger$. It holds

$$S^{-1} = \gamma_0 S^\dagger \gamma_0\,. \tag{4.38}$$

The proof of this identity is quite easy. Let us write, for convenience,

$$S = \exp(-i\Omega)\,, \tag{4.39}$$

with

$$\Omega = \frac{1}{4}\omega^{\alpha\beta}\sigma_{\alpha\beta}\,.$$

Expanding the exponential (4.39) in Taylor series, one has

$$S = \mathbb{1} + \frac{1}{1!}(-i\Omega) + \frac{1}{2!}(-i\Omega)^2 + \ldots \tag{4.40}$$

Using the relation $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ (see (3.13)), it is easy to arrive at the identity

$$\Omega^\dagger = \gamma_0\Omega\gamma_0\,. \tag{4.41}$$

This, of course, means that such a rule is valid for any power of Ω, i.e.

$$(\Omega^n)^\dagger = \gamma_0\Omega^n\gamma_0\,. \tag{4.42}$$

So, taking into account (4.42), from (4.40) one gets immediately

$$S^\dagger = \gamma_0\left[\mathbb{1} + \frac{1}{1!}i\Omega + \frac{1}{2!}(i\Omega)^2 + \ldots\right]\gamma_0\,. \tag{4.43}$$

The expression in square brackets is just $\exp(i\Omega) = S^{-1}$. Thus, we have $S^\dagger = \gamma_0 S^{-1}\gamma_0$ and the identity (4.38) is thereby proved.

One more remark is in order here. For spatial rotations, Ω is clearly Hermitian, so that the corresponding S in (4.39) is unitary. For boosts, Ω is anti-Hermitian, and S is thus Hermitian. This corresponds precisely to the known properties of the matrices Λ of Lorentz transformations.

With the basic elements of the covariant formalism at hand, let us now return briefly to the continuity equation (2.4) for the probability density and probability current. Let us see how it can be recovered from the covariant form of Dirac equation. We have

$$i\gamma^\mu\partial_\mu\psi = m\psi\,, \tag{4.44}$$

and its Hermitian conjugation becomes, upon multiplication by γ_0 from the right,

$$-i\partial_\mu \psi^\dagger \gamma_0 \gamma^\mu = m\psi^\dagger \gamma_0\,. \tag{4.45}$$

One can see that in (4.45) the expression

$$\bar{\psi} = \psi^\dagger \gamma_0 \tag{4.46}$$

emerges naturally. It is called the **Dirac conjugation** and we will use it frequently from now on. So, (4.45) is recast as

$$-i\partial_\mu \bar{\psi} \gamma^\mu = m\bar{\psi}\,. \tag{4.47}$$

Now, multiplying equations (4.44) and (4.47) by $\bar{\psi}$ and ψ from left and right, respectively, the difference of the resulting expressions gives immediately

$$\partial_\mu \left(\bar{\psi} \gamma^\mu \psi \right) = 0\,. \tag{4.48}$$

Taking into account the relations between matrices γ^μ and $\vec{\alpha}, \beta$, it is obvious that (4.48) coincides with (2.4). Thus, we have recovered the continuity equation in the covariant form

$$\partial_\mu j^\mu = 0 \tag{4.49}$$

involving a **four-current** j^μ,

$$j^\mu = \bar{\psi} \gamma^\mu \psi\,. \tag{4.50}$$

Our notation suggests that j^μ might be a four-vector under Lorentz transformations. It is indeed so. Denoting

$$j^{\mu'}(x') = \bar{\psi}'(x') \gamma^\mu \psi'(x')\,, \tag{4.51}$$

and using the transformation law (4.4), as well as the definition (4.46), one has

$$\begin{aligned} j^{\mu'}(x') &= \psi^{\dagger\prime}(x') \gamma_0 \gamma^\mu S \psi(x) \\ &= \psi^\dagger(x) S^\dagger \gamma_0 \gamma^\mu S \psi(x) \\ &= \bar{\psi}(x) \gamma_0 S^\dagger \gamma_0 \gamma^\mu S \psi(x)\,. \end{aligned} \tag{4.52}$$

So, utilizing the identity (4.38), the last expression becomes

$$j^{\mu'}(x') = \bar{\psi}(x) S^{-1} \gamma^\mu S \psi(x)\,,$$

and, taking into account (4.9), one gets finally

$$j^{\mu'}(x') = \Lambda^\mu{}_\nu j_\nu(x)\,, \tag{4.53}$$

which is the anticipated result.

Thus, we have seen that one can construct a four-vector as a bilinear form made of bispinors (in this sense, a bispinor is a “square root of a four-vector”). We will see later on that the example described above can be generalized in a systematic way; such constructions are particularly useful within the framework of field theory.

Chapter 5

C, *P* and *T*

In this chapter, we are going to examine discrete symmetries of the Dirac equation. In the title, they are shown in alphabetical order, but now we will perform a cyclic permutation and start with P. So, what is P? It is **spatial inversion**, or **space reflection**, if you want. Usually, it is also called the **parity transformation** (hence P). Such a transformation simply means

$$(x^0, \vec{x}) \longrightarrow (x^0, -\vec{x}) \,. \tag{5.1}$$

It is certainly a Lorentz transformation, since it preserves the spacetime interval $x^2 = (x^0)^2 - \vec{x}^2$. The corresponding transformation matrix is, obviously,

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{5.2}$$

Using the standard terminology (see Appendix A), spatial inversion (5.1) belongs to the set of orthochronous transformations, with $\det \Lambda_P = -1$.

Now, the question is what could be the corresponding transformation of the Dirac wave function satisfying Eq. (3.7). The relation (4.9) is quite general; it is not restricted to the continuous proper Lorentz transformations discussed in detail in the preceding chapter. So, using (4.9) and taking into account the simple structure of the matrix Λ_P, one gets immediately the conditions for the relevant matrix S_P:

$$\begin{aligned} S_P^{-1} \gamma^0 S_P &= \gamma^0 \,, \\ S_P^{-1} \gamma^k S_P &= -\gamma^k \,, \qquad k = 1, 2, 3 \,. \end{aligned} \tag{5.3}$$

Thus, the sought matrix S_P should commute with γ^0 and anticommute with any γ^k, $k = 1, 2, 3$. A solution is then clear:

$$S_P = a \cdot \gamma^0 \,, \tag{5.4}$$

where a is an arbitrary constant factor. It remains to be clarified whether (5.4) is the general solution or not. So, suppose there are two matrices R and S satisfying (5.3). It then means

$$\begin{aligned} S^{-1} \gamma^0 S &= R^{-1} \gamma^0 R \,, \\ S^{-1} \gamma^k S &= R^{-1} \gamma^k R \,, \qquad k = 1, 2, 3 \,. \end{aligned} \tag{5.5}$$

In other words, one has

$$S^{-1} \gamma^\mu S = R^{-1} \gamma^\mu R \tag{5.6}$$

for any $\mu = 0, 1, 2, 3$. Eq. (5.6) can be recast as

$$RS^{-1}\gamma^\mu (RS^{-1})^{-1} = \gamma^\mu ,$$

and this means that RS^{-1} commutes with any γ^μ, $\mu = 0, 1, 2, 3$. According to the lemma L5 from Chapter 3 this implies that

$$RS^{-1} = b \cdot \mathbb{1} , \tag{5.7}$$

where b is an arbitrary factor. Thus,

$$R = b \cdot S . \tag{5.8}$$

In view of the relation (5.8) one may conclude that (5.4) is indeed the general solution of conditions (5.3). Conventionally, we will use $S_P = \gamma_0$ henceforth.

Thus, we have established the covariance of Dirac equation under the parity transformation: if $\psi(x)$ is a solution in a given reference frame, then for $x' = (x^0, -\vec{x})$ the function

$$\psi'(x') = \gamma_0 \psi(x) \tag{5.9}$$

is the corresponding solution in the primed system. Note that this is tantamount to the statement that if $\psi(x)$ is a solution of the Dirac equation, then

$$\psi_P(x) = \gamma_0 \psi(x^0, -\vec{x}) \tag{5.10}$$

is its solution as well.

One may notice that we have achieved such a result quite easily and the parity symmetry seems to be almost automatic in the present case. In fact, as we will see later on, it is not difficult to find an example of a relativistic equation that does exhibit parity violation.

With the knowledge of the parity transformation at hand, we may now extend our previous considerations concerning bilinear forms made of Dirac spinors (cf. the discussion following the formula (4.50)). Although such a technical progress is not of immediate importance for our study of Dirac equation, it will be useful later on, within the framework of field theory (so, it will be another "rifle hanging on the wall" à la A. P. Chekhov). In any case, at the moment it may serve as a refreshing exercise for a loyal reader.

For simplicity, let us start with the expression $\bar{\psi}\psi$. Using the relation (4.38), it is easy to see that such a form is a scalar under proper Lorentz transformations. Moreover, from (5.9) it is obvious that it is invariant under spatial inversion as well. So, in this sense, $\bar{\psi}\psi$ is a true **scalar**. Next, let us consider the combination $\bar{\psi}\gamma_5\psi$. For $x' = \Lambda x$ one gets, in general,

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \psi^\dagger(x)S^\dagger\gamma_0\gamma_5 S\psi(x) = \bar{\psi}(x)\gamma_0 S^\dagger \gamma_0\gamma_5 S\psi(x) = \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x) . \tag{5.11}$$

For a proper Lorentz transformation, the generators of S are made of products of two γ-matrices, and therefore they commute with γ_5; this in turn means that $[S, \gamma_5] = 0$. Thus, we see that $\bar{\psi}\gamma_5\psi$ is invariant in such a case. On the other hand, for the spatial inversion one has $S = \gamma_0$, so that $S^{-1}\gamma_5 S = -\gamma_5$ and one thus gets

$$\bar{\psi}'(x')\gamma_5\psi'(x') = -\bar{\psi}(x)\gamma_5\psi(x) . \tag{5.12}$$

So, we end up with the conclusion that $\bar{\psi}\gamma_5\psi$ is a **pseudoscalar**.

In a similar way, one may compare the behaviour of $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma_5\psi$. We have already observed that $j^\mu = \bar{\psi}\gamma^\mu\psi$ is a four-vector (cf. (4.53)) under proper Lorentz transformations. Obviously, for spatial inversion the relation (4.53) holds equally well; this means that one has, schematically,

$$(j^0, \vec{j}) \xrightarrow{P} (j^0, -\vec{j}) , \tag{5.13}$$

i.e. j^μ is a true four-vector. For $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ one gets, after a simple manipulation,

$$j_5^\mu(x') = \psi^\dagger(x)S^\dagger\gamma_0\gamma^\mu\gamma_5 S\psi(x) = \bar{\psi}(x)S^{-1}\gamma^\mu\gamma_5 S\psi(x)\,. \tag{5.14}$$

For proper Lorentz transformations one has $[S,\gamma_5] = 0$ and (5.14) then amounts to

$$j_5^{\mu\prime}(x') = \Lambda^\mu{}_\nu j_5^\nu(x)\,.$$

For spatial inversion, $\gamma_5 S = -S\gamma_5$ and one thus gets

$$(j_5^0, \vec{j}_5) \xrightarrow{P} (-j_5^0, \vec{j}_5)\,. \tag{5.15}$$

This means that j_5^μ is a **pseudovector** (**axial vector**). Thus, in the above examples, the matrix γ_5 is responsible for the prefix “pseudo-” in the notation of the quantities in question.

Let us now proceed to the item T of our list. It denotes **time reversal** (or **time inversion**, if you want). Before examining a pertinent transformation for Dirac equation, let us return briefly to the non-relativistic Schrödinger equation mentioned in the first two chapters; it may provide us with an inspiring hint. From (1.15), (1.16) it is clear that the replacement $t \longrightarrow -t$ changes the sign of the time derivative, and the complex conjugation of the wave function does the same. Thus, one may observe that the free-particle Schrödinger equation is invariant under time reversal, in the sense that if $\psi(t,\vec{x})$ is a solution, then $\psi^*(-t,\vec{x})$ is a solution as well. So, an important point is that the transformation $t \longrightarrow -t$ is to be accompanied by the complex conjugation. This has a clear and desirable physical effect. Upon such a transformation, the energy is not changed, while the momentum changes its sign; to see this explicitly, please recall the form of a plane wave, involving the familiar factor $\exp[-i(Et - \vec{p}\cdot\vec{x})]$.

So, let us come back to our staple food, the Dirac equation. The considered transformation of spacetime coordinates is now described by means of the matrix

$$\Lambda = \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.16}$$

Motivated by the preceding considerations, we may try an Ansatz for the corresponding transformation of the Dirac wave function, defined as

$$\psi'(x') = B\psi^*(x)\,, \tag{5.17}$$

where $x' = (-x^0, \vec{x})$ and B is an invertible constant 4×4 matrix. By the way, in contrast to the preceding case of spatial inversion, (5.17) represents an **antilinear transformation**, due to the involvement of the complex conjugation — this is a well-known characteristic of the time reversal in quantum theory. So, our task is now to find the matrix B (if it exists). To this end, we start with the complex conjugation of the Dirac equation, i.e.

$$-i\gamma^{\mu *}\partial_\mu\psi^* - m\psi^* = 0\,. \tag{5.18}$$

Using (5.17), we express ψ^* as

$$\psi^*(x) = B^{-1}\psi'(x')\,, \tag{5.19}$$

and plugging this into Eq. (5.18) one gets

$$-i\gamma^{\mu *}B^{-1}\frac{\partial}{\partial x'^\lambda}\psi'(x')\frac{\partial x'^\lambda}{\partial x^\mu} - mB^{-1}\psi'(x') = 0\,. \tag{5.20}$$

Now, taking into account that

$$\frac{\partial x'^{\lambda}}{\partial x^{\mu}} = \Lambda_T{}^{\lambda}{}_{\mu}\,,$$

and multiplying Eq. (5.20) by B from the left, one obtains

$$-iB\gamma^{\mu\,*}B^{-1}\Lambda_T{}^{\lambda}{}_{\mu}\frac{\partial\psi'(x')}{\partial x'^{\lambda}} - m\psi'(x') = 0\,. \tag{5.21}$$

The condition of the covariance of Dirac equation under time reversal thus reads

$$B^{-1}\gamma^{\mu}B = -\Lambda_T{}^{\mu}{}_{\nu}\gamma^{\nu\,*}\,. \tag{5.22}$$

To work out the general relation (5.22) explicitly, one should realize that the complex conjugation of γ-matrices depends on their particular representation. In the standard representation, $\gamma^{\mu\,*} = \gamma^{\mu}$ for $\mu = 0, 1, 3$ and $\gamma^{\mu\,*} = -\gamma^{\mu}$ for $\mu = 2$. Thus, in this "household representation" the conditions (5.22) read

$$\begin{aligned} B^{-1}\gamma^0 B &= \gamma^0\,,\\ B^{-1}\gamma^1 B &= -\gamma^1\,,\\ B^{-1}\gamma^2 B &= \gamma^2\,,\\ B^{-1}\gamma^3 B &= -\gamma^3\,. \end{aligned} \tag{5.23}$$

It is easy to guess that a solution of (5.23) is

$$B = a\cdot\gamma^1\gamma^3\,, \tag{5.24}$$

where a is an arbitrary constant factor. One can also show that such a solution is unique; to prove this, one may proceed in the same way as before, in the case of the parity transformation. If we want to make (5.17) antiunitary operator, the factor a in (5.24) should be chosen so that $|a| = 1$. A conventional choice used frequently in the literature is $a = i$. Sticking to such a convention, our result can be written as

$$\psi'(x') = i\gamma^1\gamma^3\psi^*(x)\,, \tag{5.25}$$

with $x' = (-x^0, \vec{x})$.

Let us remind the reader that the concept of antiunitary vs. unitary operators is mostly due to E. P. Wigner, who is the author of a fundamental theorem on symmetries in quantum theory, which bears his name.[4]

Finally, let us take up the item C in our list. It is the so-called **charge conjugation**, and in distinction to the preceding two discrete symmetries, C is not related to any spacetime transformation. Rather, it is an **internal symmetry** and its substantial ingredient is the complex conjugation. An obvious motivation for investigating such a symmetry is the existence of free-particle solutions with positive and negative energy; one may thus naturally contemplate the possibility of a transformation turning one type of a solution into another.

We are going to start with the Ansatz

$$\psi'(x) = A\psi^*(x)\,, \tag{5.26}$$

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[4] Biographical remark: Eugene Paul Wigner (1902–1995) (originally Wigner Jenö Pál) was an eminent theorist in the field of quantum theory. Born in Hungary, he emigrated to U. S. in 1930s and received Nobel Prize in 1963 together with Maria Goeppert Mayer for the work on symmetries in nuclear and particle physics. He was a brother–in–law of Paul Dirac.